

Direct algebraic mapping transformation for decorated spin models

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Abstract

In this article we propose a general transformation for decorated spin models. The advantage of this transformation is to perform a direct mapping of a decorated spin model onto another effective spin thus simplifying algebraic computations by avoiding the proliferation of unnecessary iterative transformations and parameters that might otherwise lead to transcendental equations. Direct mapping transformation is discussed in detail for decorated Ising spin models as well as for decorated Ising-Heisenberg spin models, with arbitrary coordination number and with some constrained Hamiltonian's parameter for systems with coordination number larger than 4 (3) with (without) spin inversion symmetry respectively. In order to illustrate this transformation we give several examples of this mapping transformation, where most of them were not explored yet.

Keywords: Decoration transformation, exactly solvable modes, Ising model, Ising-Heisenberg model.

1 Introduction

Exactly solvable models is one of the most challenging topics in statistical physics and mathematical physics. Statistical physics models in general cannot be solved analytically, but only numerically. For example, Ising models with spin-1/2 or higher under external magnetic field are challenging current issues. Exact solutions were obtained only for a very limited cases. After the Onsager solution for the two-dimensional Ising model[1], several attempts to solve other similar models were performed, mainly the honeycomb lattices[2, 3]. The exact solution for the honeycomb lattice with external magnetic field was also studied by Wu[4], and the Kagome lattice was also discussed in references [5, 6]. Using the method proposed by Wu[7], Izmailian[8] obtained an exact solution for a spin-3/2 square lattice with only nearest neighbor and two-body spin interactions. Izmailian and Ananikian[8, 9] have also obtained an exact solution for a honeycomb lattice with spin-3/2. The Blume-Emery-Griffiths (BEG)[10] model for the honeycomb lattice was also investigated by Horiguchi[2], Wu [11], Tucker[12] and Urumov[13], following the standard decoration transformation[14, 15] and satisfying the Horiguchi condition[2]. The Ising model on pentagonal lattice was investigated by Waldor et al.[16] and Urumov[17]. Some exact results have been obtained with restricted parameters for spin-1 Ising model by Mi and Yang[18] using a non-one-to-one transformation[3]. Some half-odd-integer spin- S Ising models were already discussed in the literature[19]. Following this line of thought, we have recently found a family of solutions for half-odd-integer spin[20], where by means of simple projections we obtain a family of results, a particular case of which recovers the previous results found in the literature[8, 9, 19].

Several decorated Ising models have been solved using the well-known decoration transformation presented in the 1950's by M. E. Fisher[14] and Syozi[15], which has been recently generalized in reference [21] for arbitrary spin and for any *mechanical* spin, such as the classical-quantum spin model. This transformation has been widely used in the literature and, in some cases, it has been applied in several steps that introduce a number of intermediate parameters such as discussed in reference[17, 22, 23, 13]. The decoration transformation can also be applied to classical-quantum (hybrid) spin models, i.e. Ising-Heisenberg models. Several quasi-one dimensional models such as the diamond-like chain have been widely investigated in the literature [24, 25, 26, 27, 28, 29, 30], as well as two-dimensional lattice spin models by using the decoration transformation approach[31, 32, 34, 35, 36], which has been successfully applied even to three-dimensional decorated systems[37]. Another interesting application of decoration transformation was also investigated in a work by Pereira et al.[38] in which they considered a delocalized interstitial electrons on diamond-like chain and also investigated the magnetocaloric effect in a kinetically frustrated diamond chain[39]. Meanwhile,

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Strečka et al.[32] discussed the localized Ising spins and itinerant electrons in two-dimensional models, as well as two-dimensional spin-electron models with coulomb repulsion[33]. Recently, the decoration transformation approach has been also applied to spinless interacting particles, thus showing the possibility of application to interacting electron models[40]. Due to these important progresses, recently Strečka[41] discussed this transformation in more detail, following the approach presented in reference [21] for the the case of hybrid models.

On the other hand, real materials such as Heterotrimetallic $[\text{DyCuMoCu}]_\infty$ polymer[42] can be formulated as ising-Heisenberg chain models[43], as well as Dy_4Cr_4 complex[44] as decorated Ising ring. This work also was motivated due several synthesized materials, with more involving complex structure, such as the following materials: $\text{Yb}_3\text{AuGe}_2\text{In}_3$: An ordered variant of the YbAuIn structure exhibiting mixed-valent Yb behavior[45]; Density functional theory analysis of the interplay between Jahn-Teller instability, uniaxial magnetism, spin arrangement, metal-metal interaction, and spin-orbit coupling in Ca_3CoMO_6 ($\text{M} = \text{Co}, \text{Rh}, \text{Ir}$)[46]; supramolecular $\text{Co(II)-[2}\times\text{2]}$ grids: metamagnetic behavior in a single molecule[47]; magnetic ordering in Iron tricyanomethanide[48]; spin frustration in $\text{MII}[\text{C}(\text{CN})_3]_2$ ($\text{M} = \text{V}, \text{Cr}$). magnetism and neutron diffraction study[49].

In this paper we present a direct generalized transformation for a mixed or decorated spin model onto a uniform spin model, in which the main difference to the aforementioned generalized transformation[14, 21, 41] is that there is no step by step transformation. The seminal idea of decorated spin model transformation of type star-star already was emphasized and used in a particular case by Baxter[50], as well as by M. E. Fisher[53] to study the planar Ising model using dimer solution. In order to introduce a direct transformation of decorated spin model onto a uniform spin-1/2 model, we will follow the basic idea used by Baxter[50]. In order to illustrate this transformation we consider the decoration transformation displayed in figure 1:



Figure 1: The decoration transformation of the mixed spin-(1/2, S) Ising model onto its equivalent uniform spin-1/2 Ising model.

The Boltzmann factor for decorated spin model or mixed spin-(1/2, S) model (see figure 1) could be expressed by

$$w(s_1, s_2) = \sum_{\mu=-S}^S e^{J(s_1+s_2)\mu}, \quad (1)$$

in which S is any spin value larger than 1/2, whereas s_1 and s_2 are the spin values of the spin-1/2 particles. For simplicity, J represents here the spin-spin coupling in units of $-\beta = -1/kT$, in which k is Boltzmann's constant while the T means the absolute temperature. From now on we will use this convenient notation for all parameters of the Hamiltonian.

The Boltzmann factor for the effective uniform spin-1/2 model, as displayed in figure 1, could be expressed by

$$\tilde{w}(s_1, s_2) = \sum_{s=\pm\frac{1}{2}} e^{K_0+K(s_1+s_2)s}, \quad (2)$$

where K represents the spin-spin interaction parameter in units of $-\beta$, whereas K_0 is a "constant" shift energy in units of $-\beta$. The term e^{K_0} also could be understood as the Z-invariant factor[50, 53]. In eqs. (1) and (2) we assume that the spin-inversion symmetry is satisfied, i.e., the system remains invariant under reversion of all spins.

In order that both spin models become equivalent, we impose the following relation $\tilde{w}(s_1, s_2) = w(s_1, s_2)$, for the Boltzmann factors of the effective spin model and of the decorated spin model. For the spin-1/2 case we obtain two equations assuming the spin-inversion symmetry is satisfied. We obtain

$$2e^{K_0} \cosh(K/2) = \sum_{i=-S}^S \cosh(iJ), \quad (3)$$

for the configuration $\uparrow\uparrow$, whereas for the configuration $\uparrow\downarrow$ we have

$$2e^{K_0} = 2S + 1, \quad (4)$$

Therefore the constant K_0 is obtained easily from eq. (4), whereas the parameter K can be obtained from eqs. (3) and (4).

The present work aims at showing that this transformation could be easily extended for any q -leg decorated or mixed spin models, $q \in \{3, 4, \dots\}$, mapping it onto a uniform q -leg star spin-1/2 Ising models. It is organized as follows. In sec. 2 we present the generalized q -leg star-star transformation. In sec. 3 we discuss this transformation without spin-inversion symmetry. In sec. 4 it is extended to higher values of spin. In sec. 5 we discuss the transformation for quantum-classical spin and in sec. 6 we present our conclusions.

2 The generalized q -leg star-star decoration transformation

Following the transformation proposed in the previous section, we can perform a transformation assuming a general coupling for q -leg star spin model as illustrated in figure 2, under total spin-inversion invariance. The Hamiltonian for q -leg star spin model, in units of $-\beta$ may be expressed as

$$H = \sum_{j=1}^{[S]} \left(J_{2j-1} \mu^{2j-1} \sum_{i=1}^q s_i + D_{2j} \mu^{2j} \right), \quad (5)$$

where $[S]$ means the largest integer less than or equal to S , by J_j we mean the coupling coefficient of $\mu^j s_i$, whereas D_j is the coupling coefficient of μ^j with $\mu = \{-S, \dots, S\}$. Note that eq.(5) is invariant under total spin-inversion ($\sum_{i=1}^q s_i \rightarrow -\sum_{i=1}^q s_i$) and $\mu \rightarrow -\mu$.

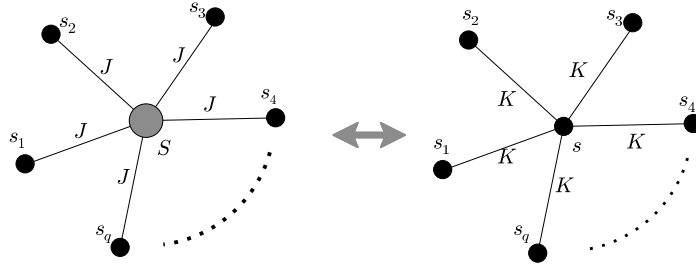


Figure 2: The q -leg star-star transformation of the mixed spin-(1/2, S) Ising model onto its equivalent uniform spin-1/2 Ising model.

Therefore the decorated q -leg star spin model Boltzmann factor could be written as

$$w(\{s_i\}) = \sum_{\mu=-S}^S \exp \left(\sum_{j=1}^{[S]} \left(J_{2j-1} \mu^{2j-1} \sum_{i=1}^q s_i + D_{2j} \mu^{2j} \right) \right), \quad (6)$$

in which by $\{s_i\}$ we mean the set of $\{s_1, s_2, \dots, s_q\}$.

On the other hand, we conveniently consider the star spin-1/2 Ising model, since models involving spin-1/2 systems could be transformed onto exactly solvable models[51]. Therefore the Boltzmann factor for uniform spin-1/2 Ising model with zero magnetic field is given by

$$\tilde{w}(\{s_i\}) = \sum_{s=\pm\frac{1}{2}} \exp \left(K_0 + K \left(\sum_{i=1}^q s_i \right) s \right). \quad (7)$$

The spin legs interacting with the central spin only depends on $\varsigma \equiv \sum_{i=1}^q s_i$, then the Boltzmann factor (6) is rewritten as

$$w(\varsigma) = \sum_{\mu=-S}^S \exp \left(\sum_{j=1}^{[S]} \left(J_{2j-1} \mu^{2j-1} \varsigma + D_{2j} \mu^{2j} \right) \right), \quad (8)$$

and their respective Boltzmann factor (7) in the effective spin model becomes

$$\tilde{w}(\varsigma) = \sum_{s=\pm\frac{1}{2}} e^{K_0 + K \varsigma s}. \quad (9)$$

Therefore the Boltzmann factors are conveniently and simply denoted by $w(\varsigma)$ and $\tilde{w}(\varsigma)$. For higher spin this notation is not any more valid, so, in that case we will consider explicitly each spin contribution.

In general, we can rewrite the result presented in the introduction as a function of Boltzmann factors of the decorated spin model, imposing the equivalence of both Boltzmann factors $\tilde{w}(s_1, s_2) = w(s_1, s_2)$, which in its turn yields

$$2e^{K_0} = w(0), \quad (10)$$

$$2e^{K_0} \cosh(K/2) = w(1). \quad (11)$$

Note that $w(0)$ and $w(1)$ must be obtained from eq. (8).

Then the solution of such algebraic system equation is expressed by

$$K_0 = \ln \left(\frac{w(0)}{2} \right), \quad (12)$$

$$K = 2 \ln \left(\frac{w(1)}{w(0)} \pm \sqrt{\left(\frac{w(1)}{w(0)} \right)^2 - 1} \right). \quad (13)$$

This transformation is equivalent to a double decoration transformation[14, 21]. The advantage of the direct mapping in this case is to avoid the unnecessary intermediate parameters (see references [14, 21]) introduced that would only make the calculation more cumbersome[17, 22] and, in some cases leading to an apparently transcendental equation.

2.1 Three-leg star-star transformation

In order to solve equations (8) and (9) for the case of three-leg star spin model, we need to assume the spin-inversion symmetry; so, we have only two configurations $\uparrow\uparrow\uparrow$ and $\uparrow\uparrow\downarrow$, which correspond to $\varsigma = 3/2$ and $\varsigma = 1/2$, respectively. Once we have two algebraic equations with two unknown parameters K_0 and K , then we are able to solve the algebraic system equations, therefore the transformation can be performed exactly for arbitrary parameter values of decorated spin models.

The unknown parameters in the effective uniform spin-1/2 model will be expressed in terms of all arbitrary parameters of the decorated spin model, assuming that both models are equivalent which means $\tilde{w}(\varsigma) = w(\varsigma)$, thus, we have,

$$2e^{K_0} \cosh(K/4) = w(1/2), \quad (14)$$

$$2e^{K_0} \cosh(3K/4) = w(3/2), \quad (15)$$

thus, the solution of the algebraic system equations can be written explicitly by

$$K = 2 \ln \left(\frac{w(3/2)}{w(1/2)} + 1 \pm \sqrt{\left(\frac{w(3/2)}{w(1/2)} + 1 \right)^2 - 4} \right) - 2 \ln(2), \quad (16)$$

$$K_0 = \frac{1}{2} \ln \left(\frac{w(1/2)^3}{3w(1/2) + w(3/2)} \right). \quad (17)$$

Once again this transformation is equivalent to a double transformation (something like as star-triangle-star transformation). By using a star-star direct transformation[50], we avoid the introduction of unnecessary intermediate parameters (such as the intermediate parameters to represent the triangle structure system) customary in the literature (e.g. [17, 22]) making the mapping more easy to manipulate.

2.2 Four-leg star-star transformation

Another important transformation is the four-leg decorated spin model, in which there are three spin configurations for the legs: $\uparrow\uparrow\uparrow\uparrow$, $\uparrow\uparrow\uparrow\downarrow$ and $\uparrow\uparrow\downarrow\downarrow$. Under total spin-inversion symmetry, any permutations and inversions of spin always fall into one of these three configurations. Using the notation $\varsigma = s_1 + s_2 + s_3 + s_4$, these three configuration correspond just to $\varsigma = 0, 1$ and 2 , respectively. Assuming both Boltzmann factors are equivalent $\tilde{w}(\varsigma) = w(\varsigma)$, the algebraic systems equation becomes,

$$2e^{K_0} = w(0), \quad (18)$$

$$2e^{K_0} \cosh(K/2) = w(1), \quad (19)$$

$$2e^{K_0} \cosh(K) = w(2). \quad (20)$$

The first two equations eqs. (18) and (19) are identical to those found for the case of two-leg transformation which is given by eq. (10) and (11) respectively. For the case of four-leg star-star transformation, we have one additional equation given by (20), but similarly to the previous case, there are only two unknown parameters. In order to satisfy completely the algebraic system of equations we need to impose the following additional relation between Boltzmann factors of the decorated spin model, yielded by the manipulation of eqs. (18-20),

$$w(0)w(2) + w(0)^2 = 2w(1)^2. \quad (21)$$

This means that, in general, at most two parameters of a decorated spin model could be constrained.

It is interesting to highlight that eq.(21) was obtained only using algebraic manipulation in order to satisfy the algebraic system of equations given by (18)-(20). Surprisingly the relation (21) represent nothing but the special case of the *free fermion condition* of 8-vertex model[51] formulated for four-leg star. The 8-vertex model configuration displayed in figure 3 can be compared with eq.(21) by the following relations $\omega_1 = w(2)$, $\omega_2 = \omega_3 = \omega_4 = w(0)$ and $\omega_5 = \omega_6 = \omega_7 = \omega_8 = w(1)$.

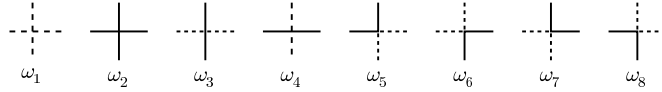


Figure 3: A line representation of the eight-vertex[51] model formulated for the four-leg star.

In principle, some mixed spin with four-leg star and constrained parameters can be mapped onto an exactly solvable rectangular Ising model [51]. In what follows we discuss this mapping for some particular case.

For integer values of the central spin, such as spin-1, eq.(21) lead us only to a trivial solution ($J_1 = 0$). For spin-2, the Boltzmann weights are obtained from eq. (8), then eq. (21) reduce to

$$2t^{16} (y^3x - 1)^2 (xy^7 - 1)^2 (y^6x^2 + y^3x + 1)^2 r^4 + t^{15} (xy^4 - 1)^4 (xy^4 + 1)^4 r^3 + x^2y^{14} (xy - 1)^4 = 0, \quad (22)$$

where $x = \exp(J_1)$, $y = \exp(J_3)$, $r = \exp(D_2)$ and $t = \exp(D_4)$. Thus we can verify that eq.(22) becomes a quartic equation in relation to the variable r , and their coefficients are all non-negatively defined, therefore once again we only obtain a trivial solution for $J_1 = 0$ and $J_3 = 0$. We expect this property should predominate for higher order integer spins, in accordance with those discussed in reference [20], where the integer spin cannot be mapped onto a spin-1/2 Ising model.

However for half-odd-integer central spin we obtain some non-trivial solutions.

- For spin-3/2 the condition (21) becomes

$$r^{10} (y^7x - 1)^2 (xy^{13} + 1)^2 (xy^{13} - 1)^2 (y^7x + 1)^2 (x^2y^{14} + 1)^2 = 0, \quad (23)$$

with $x = \exp(J_1/2)$, $y = \exp(J_3/8)$ and $r = \exp(D_2/4)$. From eq.(23) we obtain a non-trivial results, recovering the previous results obtained by a different approach, for more detail see reference [8, 20].

- For spin-5/2 the eq. (21), may be expressed in a similar way to the previous cases. After some tedious algebraic manipulation we have

$$0 = r^3 t^{39} (x^4 y^{76} z^{1684} - 1)^2 (xy^{49} z^{1441} - 1)^2 + r^2 t^{34} (x^2 y^{62} z^{1562} - 1)^2 (x^3 y^{63} z^{1563} - 1)^2 + x^2 y^{98} z^{2882} (x^2 y^{14} z^{122} - 1)^2 (xy^{13} z^{121} - 1)^2, \quad (24)$$

where $x = \exp(J_1/2)$, $y = \exp(J_3/8)$, $z = \exp(J_5/32)$, $r = \exp(D_2/4)$ and $t = \exp(D_4/16)$. The eq. (24) is a cubic equation in relation to the variable r and all coefficients of eq.(24) are non-negative, therefore there is no positive solution for r , unless all coefficients becomes simultaneously zero. In order to find the solutions of eq. (24) we chose the following possibility, let us assume that $x^4 y^{76} z^{1441} = 1$ from the coefficient of term r^3 and $x^2 y^{62} z^{15632} = 1$ from the coefficients of r^2 , thus we have $y = z^{-30}$ and $x = y^{-31} z^{-781} = z^{149}$. Satisfying this conditions the eq. (24) is identically zero. This expression in terms of the Hamiltonian's parameter becomes as $J_3 = -\frac{15}{2}J_5$ and $J_1 = \frac{149}{32}J_5$, thus the Boltzmann factor given by eq. (8) reduce to

$$w(\varsigma) = \sum_{\mu=-5/2}^{5/2} \exp \left(J_5 \left(\frac{149}{16}\mu - \frac{15}{2}\mu^3 + \mu^5 \right) \varsigma \right) = \sum_{\mu=-5/2}^{5/2} e^{J_5 \frac{51}{16}\sigma(\mu)\varsigma}, \quad (25)$$

where $\sigma(\mu)$ is defined by

$$\sigma(\mu) = \frac{149}{120}\mu - \mu^3 + \frac{2}{15}\mu^5, \quad (26)$$

with μ we represent the Ising spin-5/2. Note that $\sigma(\mu)$ only takes special values $\{\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\}$ when μ takes $\{5/2, 3/2, 1/2, -1/2, -3/2, -5/2\}$ respectively. Using similar process for other combination of constrained parameters, we obtain additional solutions from eq. (24), recovering all the three solutions obtained previously in reference [20] using a different mapping (for detail see the eq. (25) of reference [20]).

Following the present mapping, we could recover the mapping for higher half-odd-integer spins obtained previously in reference [20], where was used a different approach, furthermore we should probably obtain even additional results to those already found in [20], as was shown here for spin-5/2.

2.3 General condition for q -leg star-star transformation

For general q -leg Ising spin star-star transformations with central spin S it is possible to obtain the solution for arbitrary q .

- In order to satisfy the condition of star-star transformation, first we consider the case of even values of q ($q \geq 4$). We have two unknown parameters and $q/2$ algebraic equations. Therefore we must have $(q/2 - 2)$ conditions to be satisfied for the decorated q -leg star spin model, which read

$$2w(\frac{r}{4})^2 = w(0) \left(w(\frac{r}{2}) + w(0) \right); \text{ for } \frac{r}{2} : \text{ even}, \quad (27)$$

$$2w(\frac{r}{4} - \frac{1}{2})w(\frac{r}{4} + \frac{1}{2}) = w(0) \left(w(\frac{r}{2}) + w(1) \right); \text{ for } \frac{r}{2} : \text{ odd}, \quad (28)$$

with $r = \{4, 6, 8, \dots, q\}$. It is worth to notice that this condition depends on the even or odd character of $r/2$.

- On the other hand, for the case of q odd (for $q \geq 5$), we still have two unknown parameters and $([q/2] - 2)$ algebraic system equations, so the parameters of the original q -leg star Ising spin model must satisfy the following conditions,

$$w(\frac{r}{2} - 1) \left(w(\frac{r}{2} - 1) + w(\frac{r}{2} - 3) \right) = w(\frac{r}{2} - 2) \left(w(\frac{r}{2}) + w(\frac{r}{2} - 2) \right), \quad (29)$$

where $r = \{5, 7, 9, \dots, q\}$. For the case of q odd we only have one kind of relation for a given odd r .

As we can see, the number of constrained parameters increases with the number of legs or the coordination number, whereas the maximum number of coupling parameter increases with the spin S . The condition for any arbitrary q -leg spin are identical to those of the $(q - 2)$ -leg decoration transformation plus one additional condition that only appears when q -leg spin is considered. In other words, all conditions on $(q - 2)$ -leg are valid for q -leg star plus one new additional condition that involves $w(\frac{q}{2})$ as displayed by eqs.(27), (28) and (29).

This transformation should correspond to the double star-polygon-star transformation proposed in reference [21], where the polygons involve long-range interactions. However, the decoration transformation proposed in [14, 15, 21] leads to a cumbersome coupling, whereas the direct transformation proposed here just needs to satisfy the conditions (27) and (28) for q even, and the condition (29) must be satisfied for q odd.

3 Star-star transformation without spin-inversion symmetry

The transformation previously discussed could be easily extended for the q -leg star spin model without spin-inversion symmetry too. For a decorated or mixed spin model transformed onto a uniform spin-1/2 q -leg star Ising model with external magnetic field, its decorated q -leg star Boltzmann factor can be written in a similar way as in eq. (6), in which the spin legs interacting with decorated spin depend only on $(\varsigma = \sum_{i=1}^q s_i)$ which we can denote for simplicity only by ς , thus yielding

$$w(\varsigma) = \sum_{\mu=-S}^S \exp \left(\sum_{j=1}^{2S} (J_j \mu^j \varsigma + D_j \mu^j - B \varsigma / q) \right), \quad (30)$$

with J_j and D_j are coupling parameters, and B represents the external magnetic field on the legs. The effective Boltzmann factor for an uniform star Ising spin model is expressed by

$$\tilde{w}(\varsigma) = \sum_{s=\pm \frac{1}{2}} e^{K_0 + K \varsigma s - h \varsigma / q - h_0 s}. \quad (31)$$

where K_0 is a constant energy, K is the coupling term, with h and h_0 being the external magnetic field for the legs and central spin respectively.

A particular case of this transformation is the two-leg ($q = 2$) decoration transformation without spin-inversion and with $h_0 = h$ in eq. (31). Therefore assuming the Boltzmann factor (30) is equivalent to (31), we have $\tilde{w}(\varsigma) = w(\varsigma)$. For this case we have three equations and three unknown parameters that must satisfy the following relations

$$a = \frac{w_0 c}{1 + c^2}, \quad (32)$$

$$b = \frac{c \left(w_{-1} - \sqrt{w_1 w_{-1} - w_0^2} \right)}{w_0}, \quad (33)$$

$$c = \frac{w_0^2 \pm \delta \sqrt{-w_0^2 + w_1 w_{-1}}}{w_0^2 + w_{-1} \delta}, \quad (34)$$

in which, for simplicity, the Boltzmann factor is denoted by $w_\varsigma \equiv w(\varsigma)$, and $\delta = w_{-1} - w_1$. The variables are defined as $a = \exp(K_0)$, $b = \exp(K/2)$ and $c = \exp(-h/2)$.

Another transformation that we consider is the three-leg star-star transformation. For this case we have four equations and four unknown parameters. This transformation is related to the 8-vertex model on the honeycomb lattice such as discussed by Lin and Wu[52]. The schematic representation of 8-vertex model for the honeycomb lattice is given in figure 4.

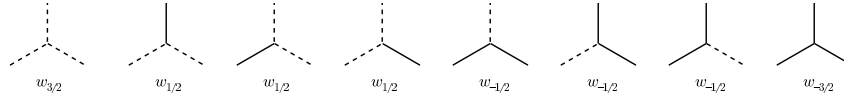


Figure 4: A line representation of 8-vertex configuration for three-leg star.

Therefore in a similar way to that of the honeycomb lattice we may express the solutions in terms of the Boltzmann factors as follows,

$$e^{-\frac{2}{3}h} = \frac{w_{3/2}w_{-1/2} - w_{1/2}^2}{w_{-3/2}w_{1/2} - w_{-1/2}^2}, \quad (35)$$

$$\cosh^2(K/2) = \frac{1}{4} \frac{(w_{3/2}w_{-3/2} - w_{1/2}w_{-1/2})^2}{(w_{3/2}w_{-1/2} - w_{1/2}^2)(w_{-3/2}w_{1/2} - w_{-1/2}^2)}, \quad (36)$$

$$\sinh(h_0) = \frac{\sinh(K/2)(w_{3/2}w_{-1/2}^3 - w_{-3/2}w_{1/2}^3)}{(w_{3/2}w_{-1/2} - w_{1/2}^2)(w_{-3/2}w_{1/2} - w_{-1/2}^2)}, \quad (37)$$

$$e^{4K_0} = \frac{(w_{3/2}w_{-1/2} - w_{1/2}^2)(w_{-3/2}w_{1/2} - w_{-1/2}^2)}{16 \sinh^4(K/2)}. \quad (38)$$

As we can verify, the effective spin model parameters always may be expressed in terms of Boltzmann factors. A particular case of this transformation could be the mixed spin- $(1/2, S)$ Ising model on the honeycomb lattice, which can be transformed onto the spin- $1/2$ Ising model with external magnetic field on honeycomb lattice, such as considered by Azaria-Giacomini[5] and Wu[4, 7, 11], where this model is related to 8-vertex model as displayed in fig.4.

If we consider a particular case of the decorated spin model with uniform external magnetic field on the honeycomb lattice, i.e. $h_0 = h$. This leads to constrained parameters, the constraints of which could be written as in previous cases, even though more involved.

Finally the extension for coordination number larger than 3 can be performed straightforwardly, although the conditions of Boltzmann factors will become more involved expressions.

4 Star-star transformation for higher spin

Another interesting case that worth to comment is when the q -leg star-star transformation has spin larger than spin- $1/2$. Certainly, this kind of model can be extended in a similar way as in [21]; however, the decoration transformation is subject to more conditions (and thus more constrained parameters appears) for its validity.

The Hamiltonian for the decorated spin model with q -legs could have a similar treatment to that of sec.2, thus we may write

$$H = \sum_{i=1}^{2S} \sum_{r=1}^q \left(\sum_{j=1}^{S_0} J_{i,j} \mu_r^i \mu^j - \frac{1}{q} D_i \mu_r^i \right) - \sum_{i=1}^{2S_0} B_i \mu^i, \quad (39)$$

in which we use the notation $J_{i,j}$ for the parameter of the Hamiltonian, which corresponds to the coupling coefficients of the term $\mu_r^i \mu^j$, whereas by D_i we represent the coupling coefficient of term μ_r^i , and the last term B_i means the coupling coefficient of the term μ^i , with $\mu = \{-S, \dots, S\}$.

Following the same process developed in sec.2 and illustrated in fig.2, the Hamiltonian of the intermediate mixed spin model becomes

$$H' = \sum_{i=1}^{2S} \sum_{r=1}^q \left(M_i \mu_r^i \sigma - \frac{1}{q} D'_i \mu_r^i \right) - h\sigma + M_0, \quad (40)$$

in which M_i represents the parameter of $\mu_r^i \sigma$, whereas D'_i are the coefficients of the term μ_r^i , with h being the external magnetic field strength and M_0 corresponds to constant energy terms. Using the direct decoration transformation we can map the system with Hamiltonian (39) onto another effective system with Hamiltonian (40).

It is worth to notice that the Boltzmann factor of higher order spins does not depend only on $\varsigma = \sum_{i=1}^q \mu_i$, but also depends on each spin μ_i ; consequently, we need to express explicitly the Boltzmann factor in terms of each spin μ_i .

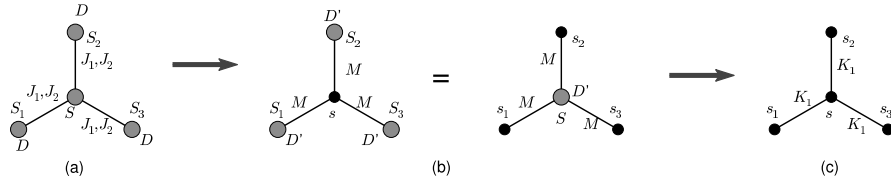


Figure 5: Uniform spin-1 star transformation onto its uniform spin-1/2 star

As an illustrative example let us consider the uniform spin-1 ($S = S_0 = 1$) with $q = 3$, as displayed in fig.5(a), transforming onto the effective model described by left side of fig.5(b) ($S = 1$, $\sigma = \pm 1/2$). Under spin inversion symmetry the Boltzmann factor of decorated spin model becomes,

$$W(\mu_1, \mu_2, \mu_3) = \sum_{\mu=-1}^1 \exp \left(J_{11}(\mu_1 + \mu_2 + \mu_3)\mu + J_{22}(\mu_1^2 + \mu_2^2 + \mu_3^2)S^2 - D_2(\mu_1^2 + \mu_2^2 + \mu_3^2)/3 - B_1\mu^2 \right), \quad (41)$$

whereas the Boltzmann factor for effective spin models is given by

$$W'(\mu_1, \mu_2, \mu_3) = \exp \left(M_0 + -D'_2(\mu_1^2 + \mu_2^2 + \mu_3^2)/3 \right) \sum_{s=\pm\frac{1}{2}} e^{M_1(\mu_1+\mu_2+\mu_3)s}. \quad (42)$$

Using the Boltzmann factor of this elementary cell, we are able to reproduce i.e. the honeycomb lattice Ising model with spin-1 and mixed spin-(1,1/2) as illustrated in fig.5(a) and fig.5(b) respectively. Imposing the relation $W'(\mu_1, \mu_2, \mu_3) = W(\mu_1, \mu_2, \mu_3)$, we have six configurations and three unknown parameters to be determined for the effective spin-(1,1/2) star. In analogy to the previous case we must have three identities that must be satisfied by the Boltzmann factors.

$$W(1, 1, 1)W(0, 0, 0) = W(0, 1, 0)W(1, 0, -1), \quad (43)$$

$$2W(1, 1, 0)W(1, 0, 0) = (W(1, 1, 1) + W(1, 1, -1))W(0, 0, 0), \quad (44)$$

$$2W(1, 0, 0)^2 = (W(1, 1, 0) + W(1, -1, 0))W(0, 0, 0), \quad (45)$$

For the Hamiltonian (39) the previous eqs. (43-45) are all equivalent, leading just to one relation for the decoration transformation the parameters of Boltzmann factor which satisfy the following relation

$$\exp(J_{22}) = \cosh(J_{11}), \quad (46)$$

which is also known as Horiguchi's condition[2], obtained using the standard decoration transformation[14, 15].

For the dual lattice in fig.5(b) (right side), the Boltzmann factor is given by

$$\tilde{w}'(s_1 + s_2 + s_3) = e^{M_0} \sum_{\mu=-1}^1 \exp(M_1(s_1 + s_2 + s_3)\mu - D'\mu^2), \quad (47)$$

and it can be expressed in terms of (41) as

$$\tilde{w}'(3/2) = \frac{1}{2}W(0, 0, 0) + W(1, 1, 1), \quad (48)$$

$$\tilde{w}'(1/2) = \frac{1}{2}W(0, 0, 0) + W(1, 1, -1). \quad (49)$$

The Hamiltonian of 3-leg star effective spin model could be expressed as

$$\tilde{H} = K_0 + K(s_1 + s_2 + s_3)s. \quad (50)$$

The decoration transformation will be applied in a similar way to the previous transformation discussed in section 2 (for details see fig. 2), where the spins on the legs have same values for both models. Then the Boltzmann factor is given by (9).

Performing a further direct transformation $\tilde{w}'(\varsigma) = \tilde{w}(\varsigma)$, as illustrated in fig. 5, we obtain the results given by eqs. (16) and (17) for K and K_0 respectively.

Here we showed how the direct transformation could be applied in just two steps only (see fig. 5), rather than in five steps via the standard decoration transformation[13], we verify that constrained parameter given by eq. (2) of reference[13] is identical to our eq.(46).

5 Decoration transformation for classical-quantum spin models

The transformation presented in section 2 also can be extended for classical-quantum (hybrid) spin models such as Ising-Heisenberg models, following a similar approach proposed recently by Strecka[41]. Here we show how this transformation can be used for a particular kind of lattice without losing its general properties.

5.1 Hybrid-star decoration transformation

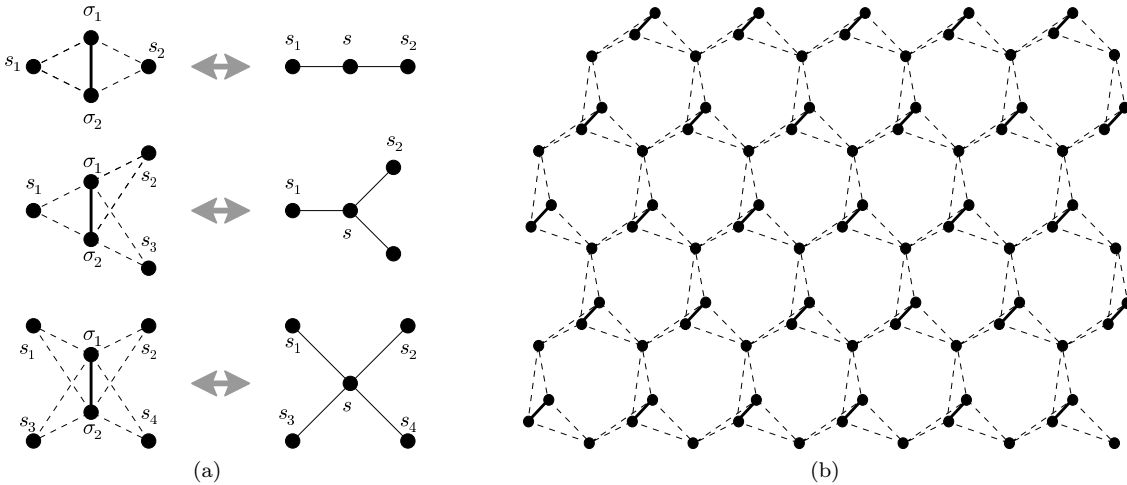


Figure 6: (a) Some examples of hybrid-star decoration transformation. The thick solid line represents a Heisenberg-like interaction whereas the dashed line and thin solid line represents the Ising interaction. In (b) we display a schematic representation of a lattice, assembled by a second transformation illustrated in (a).

As a first case let us consider the hybrid-star decoration transformation displayed in figure 6(a), in which solid thick lines represent a Heisenberg-like interaction whereas dashed and solid thin lines represent the Ising interaction. Thus the Boltzmann factor may be expressed by

$$w(\{\sigma\}) = \text{tr}_{\{\sigma\}} \exp \left(H^{XXZ}(\sigma_1, \sigma_2) + J_2(\sigma_1^z + \sigma_2^z)(s_1 + s_2 + \dots + s_q) \right), \quad (51)$$

where $H^{XXZ}(\sigma_1, \sigma_2) = J_1(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y) + J_z \sigma_1^z \sigma_2^z$, and J_1 , J_2 and J_z are interacting parameters of the Hamiltonian.

In order to calculate the trace in eq. (51) first we diagonalize the Hamiltonian (51), and then we obtain the Boltzmann factor which reads

$$w(\varsigma) = 2e^{J_z/4} \cosh(J_2 \varsigma) + 2e^{-J_z/4} \cosh(J_1/2). \quad (52)$$

However the effective Boltzmann factors is still expressed by eq. (9). This transformation can be applied to several types of lattice, mainly in one and two dimensional models. The quasi-two-dimensional Ising-Heisenberg model represented in fig. 6(b) can be transformed onto an exactly solvable two-dimensional Ising model using the second transformation illustrated in fig. 6(a).

Now let us consider another example, a three-leg hybrid-star transformation, where the decoration consists of three spins forming a triangle, in which the internal interaction legs could be either Ising type or Heisenberg interactions, whereas the external legs are necessarily of the Ising type. This transformation is displayed in figure 7. The limiting case of the transformation illustrated in fig.7 (the Ising coupling) is also known as extended-reduced lattice[53].

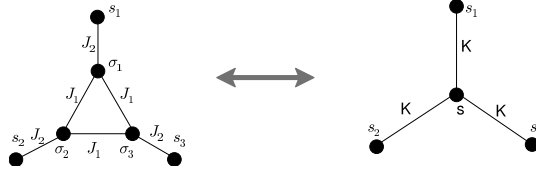


Figure 7: 3-leg hybrid-star decoration transformation onto uniform spin model.

The Boltzmann factors of the decorated hybrid spin model reads

$$w(\{s\}) = \text{tr}_{\{\sigma\}} \exp \left(H^{XXZ}(\sigma_1, \sigma_2, \sigma_3) + J_2(\sigma_1^z s_1 + \sigma_2^z s_2 + \sigma_3^z s_3) \right), \quad (53)$$

where $H^{XXZ}(\sigma_1, \sigma_2, \sigma_3) = \sum_{i=1}^3 \{ J_1 (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) + J_z \sigma_i^z \sigma_{i+1}^z \}$ with $\sigma_4 = \sigma_1$.

The triangle cell (decorated) could be expressed as Heisenberg coupling, and as expected, we obtain two configurations for their legs, in correspondence to the configurations $\uparrow\uparrow\uparrow$ and $\uparrow\uparrow\downarrow$, so that, in terms of ς we have $\varsigma = 3/2$ and $1/2$ respectively. Therefore, we obtain the following Boltzmann factors,

$$w(1/2) = 2 \left(e^{\frac{3}{4}J_z} + e^{-\frac{1}{4}J_z - \frac{1}{2}J_1} \right) \cosh \left(\frac{J_2}{4} \right) + 2e^{\frac{1}{4}(-J_z + J_1)} \times, \quad (54)$$

$$\left(e^{-\frac{1}{4}J_2} \cosh \left(\frac{1}{2} \sqrt{J_2^2 + \frac{9}{4}J_1^2 + J_1 J_2} \right) + e^{\frac{1}{4}J_2} \cosh \left(\frac{1}{2} \sqrt{J_2^2 + \frac{9}{4}J_1^2 - J_1 J_2} \right) \right)$$

$$w(3/2) = 2e^{\frac{3}{4}J_z} \cosh \left(\frac{3J_2}{4} \right) + 2e^{-\frac{1}{4}J_z} \left(e^{J_1} + 2e^{-\frac{1}{2}J_1} \right) \cosh \left(\frac{J_2}{4} \right). \quad (55)$$

This hybrid-star transformation could be applied to find the exact solution of Ising-Heisenberg type models i.e. the 3-9 (triangle-nonagon) lattice as displayed in figure 8(a), where in its triangle cell we have Heisenberg interaction, whereas in its nonagon cell we have alternating Ising-Ising-Heisenberg coupling. This lattice could be mapped onto a honeycomb Ising model[4, 52]. In figure 8(b) we display the Ising-Heisenberg model on the 3-12 (triangle-dodecagon) lattice, that can be solved exactly using the hybrid-star transformation, where once again we have Heisenberg coupling on triangle cell, and the dodecagon cell has Ising-Ising-Heisenberg coupling. Detailed discussion about the thermodynamics properties of these lattice could be analyzed, but this issue is beyond the scope of this work.

A general expression for hybrid decorated spin model can also be discussed, where hybrid decoration particle interaction could be expressed in general by the Hamiltonian $H^c(\{\sigma\})$ (such as Heisenberg interactions), in which $\{\sigma\}$ stands for the set of spin operators that plays the role of central *mechanical* spin, while the interaction of

central decorated spin with their legs could be given in general by $H^l(\{\sigma\}, \{s\})$, in which $\{s\}$ is the set of Ising spins on the legs. Therefore the general Boltzmann factor of hybrid spin model could be expressed by

$$w(\{s\}) = \text{tr}_{\{\sigma\}} \exp(H^c(\{\sigma\}) + H^l(\{\sigma\}, \{s\})). \quad (56)$$

The hybrid-star transformation is equivalent to the hybrid-polygon-star transformation[21, 41]; certainly, the parameters acting on polygons could make the transformation an involving task.

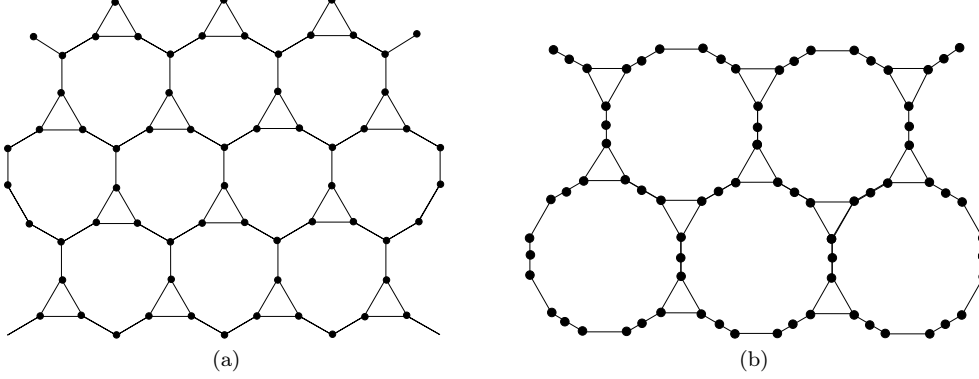


Figure 8: Two possible hybrid lattice, that could be mapped onto a honeycomb Ising model: (a) the triangle-nonagon (3-9) Ising-Heisenberg model; and (b) the triangle-dodecagon (3-12) Ising-Heisenberg model.

Alternatively the star-star transform can be generalized even when the legs interact, as we can see in figure 10(b), where the transformation not necessarily involves star-like cells. This transformation could be useful to perform a direct transformation, such as the square-hexagonal (4-6) lattice[54], square octogonal (4-8) lattice[55], and the pentagonal lattice[17].

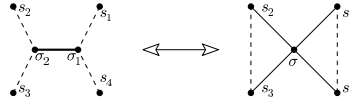


Figure 9: The alternative hybrid-star like transformation onto its equivalent uniform spin-1/2 Ising model.

The left side Boltzmann's factor of figure 9 may be written as

$$w(\{s\}) = \text{tr}_{\{\sigma\}} \left\{ \exp(H^{XXZ}(\sigma_1, \sigma_2) + J[\sigma_1(s_1 + s_2) + \sigma_2(s_3 + s_4)]) \right\}, \quad (57)$$

with σ_1 and σ_2 being Pauli matrix.

On the other hand, the Boltzmann factor of transformed plaquette (right side of figure 9) is given by

$$\tilde{w}(\{s\}) = \sum_{\sigma=\pm 1/2} \exp(K_0 + K_1\sigma(s_1 + s_2 + s_3 + s_4) + K_2(s_1s_2 + s_3s_4)), \quad (58)$$

where K_0 is a constant shift energy, whereas K_1 is the interaction parameter between the internal Ising spin σ and each spin $\{s_i\}$ and finally K_2 is the coupling term between $\{s_i\}$, with $\sigma = \pm 1/2$ and $s = \pm 1/2$.

Imposing the condition $w(\{\tau\}) = \tilde{w}(\{\tau\})$, for arbitrary $\{s_i\}$, we obtain only four nonequivalent configurations, namely, $\{s_1, s_2, s_3, s_4\} = \{+, +, +, +\}$, $\{+, +, +, -\}$, $\{+, +, -, -\}$ and $\{+, -, +, -\}$. Any other permutation or spin inversion falls into one of these configurations. Thus the Boltzmann factors reads

$$\omega_1 = w(+, +, +, +) = 2e^{K_0 + K_2/2} \cosh(K_1), \quad (59)$$

$$\omega_2 = w(+, -, +, -) = 2e^{K_0 - K_2/2}, \quad (60)$$

$$\omega_3 = w(+, +, -, -) = 2e^{K_0 + K_2/2}, \quad (61)$$

$$\omega_5 = w(+, -, +, -) = 2e^{K_0} \cosh(K_1/2). \quad (62)$$

In order to solve the above equation consistently, the algebraic equation must satisfy the following relation

$$2\omega_5^2 = (\omega_1 + \omega_3)\omega_2. \quad (63)$$

After performing some algebraic manipulation of eq. (59-62), we obtain the magnitudes of the effective interactions this

$$e^{2K_0} = \frac{\omega_3\omega_2}{4}, \quad (64)$$

$$e^{K_1} = \frac{\omega_1}{\omega_3} \pm \sqrt{\left(\frac{\omega_1}{\omega_3}\right)^2 - 1}, \quad (65)$$

$$e^{K_2} = \frac{\omega_3}{\omega_2}. \quad (66)$$

This results is equivalent to that obtained by Urumov[17], using a standard decoration transformation, for more detail the reader is referred to reference [17], and it can be compared with our results, showing how the direct transformation avoids the intermediate transformation.

It is worth to notice that the relations (63)-(66) could be valid for any arbitrary spin- S_1 and spin- S_2 instead of σ_1 and σ_2 respectively in eq. (57). With higher order coupling term on hybrid system, such that satisfy the spin-inversion symmetry.

5.2 Hybrid-hybrid transformation

In general it is still possible to extend the decoration transformation to hybrid-hybrid transformation. This kind of transformation could be for the direct mapping of some hybrid model, such as given by the Hamiltonian (56) onto another hybrid model with different topological structure. Physically, this could help the understanding of physical properties of two different hybrid models (see fig. 10(a)). Thus the Boltzmann factor of the effective hybrid spin model may be expressed by

$$\tilde{w}(\{s\}) = \text{tr}_{\{\tau\}} \exp \left(K_0 + \tilde{H}^c(\{\tau\}) + \tilde{H}^l(\{\tau\}, \{s\}) \right), \quad (67)$$

where $\tilde{H}^c(\{\tau\})$ is the Hamiltonian of the central *mechanical* spin, $\{\tau\}$ are the spin operators of effective lattice that interacts inside the *mechanical* spin, while the Hamiltonian $\tilde{H}^l(\{\tau\}, \{s\})$ represents the interaction of the central *mechanical* spin and its legs with spins $\{s\}$.

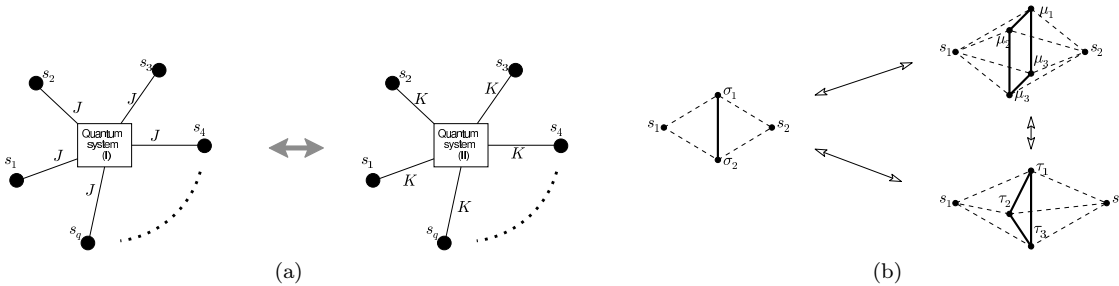


Figure 10: (a) The hybrid-hybrid decorated transformation, onto its equivalent hybrid system. (b) Example of hybrid-hybrid decoration transformation.

As an illustrative example that we consider, could be the displayed in figure 10 (b), the Boltzmann factor for the decorated spin model becomes

$$w(\{s\}) = \text{tr}_{\{\sigma\}} \left(e^{H^{XXZ}(\sigma_1, \sigma_2) + J_2(\sigma_1^z + \sigma_2^z)(s_1 + s_2)} \right), \quad (68)$$

whereas the effective Boltzmann factor will be given by eq. (52)

$$\tilde{w}(\varsigma) = 2e^{K_z/4} \cosh(K_2\varsigma) + 2e^{-K_z/4} \cosh(K_1/2). \quad (69)$$

Imposing that eq.(68) and (69) be equivalent we obtain the following relations,

$$e^{K_z/4} = \frac{w(1) - w(0)}{2(\cosh(K_2) - 1)}, \quad (70)$$

$$K_1 = 2\text{arccosh}\left(e^{K_z/4}\left(\frac{w(0)}{2} - e^{K_z/4}\right)\right). \quad (71)$$

Note that the parameter K_2 is an independent parameter in this transformation.

The hybrid-hybrid transformation is equivalent to the standard hybrid-polygon-hybrid decoration transformation [21, 41]; clearly, the algebraic manipulations involved in the hybrid-hybrid transformation are much easier to perform.

6 Conclusions

In this paper we present a direct transformation for a general decorated spin model. We have discussed that the advantage of this transformation is that of avoiding the proliferation of unnecessary intermediate parameter which only makes the algebraic calculation cumbersome. We have discussed the q -leg star-star transformation with any central *mechanical* spin and spin-1/2 particles on their legs, thus finding that the transformation will be possible for $q \geq 4$, only if the decorated spin model satisfy the conditions given in eqs. (27)-(29) and spin-inversion symmetry. When spin-inversion symmetry is not satisfied, this conditions becomes a more involving relation. The case of higher order spins has been also discussed, and we show that the expression of parameter constraints becomes more cumbersome. Finally the extension of decoration transformation to classical-quantum (hybrid) spin models has been discussed as well, in which several decorated hybrid spin models could be mapped onto other hybrid spin models with different topology. All transformation discussed above are illustrated by several examples and most of these models were not explored yet, therefore it could be interesting to discuss this models in order to study its thermodynamic properties.

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